Quantificational Logic Equivalences

In the principles below, we use $\varphi(x)$ to indicate any wff with x as a free variable

Quantifier Negation:

1a. $\neg\forall x\ \varphi(x) \Leftrightarrow \exists x\ \neg\varphi(x)$
1b. $\neg\exists x\ \varphi(x) \Leftrightarrow \forall x\ \neg\varphi(x)$

Null Quantification: (x does not occur as a free variable in φ)

2a. $\forall x\ \varphi \Leftrightarrow \varphi$
2b. $\exists x\ \varphi \Leftrightarrow \varphi$

Replacing Bound Variables: ($\varphi(y)$ is the wff that results by substituting y for every free variable x in $\varphi(x)$, where y does not already occur as a free variable in $\varphi(x)$)

3a. $\forall x\ \varphi(x) \Leftrightarrow \forall y\ \varphi(y)$
3b. $\exists x\ \varphi(x) \Leftrightarrow \exists y\ \varphi(y)$

Swapping Quantifiers of Same Type

4a. $\forall x \forall y\ \varphi(x,y) \Leftrightarrow \forall y \forall x\ \varphi(x,y)$
4b. $\exists x \exists y\ \varphi(x,y) \Leftrightarrow \exists y \exists x\ \varphi(x,y)$

Aristotelean Square of Opposition:

5a. $\neg\forall x\ (\varphi(x) \rightarrow \psi(x)) \Leftrightarrow \exists x\ (\varphi(x) \wedge \neg\psi(x))$
5b. $\neg\exists x\ (\varphi(x) \wedge \psi(x)) \Leftrightarrow \forall x\ (\varphi(x) \rightarrow \neg\psi(x))$

Quantifier Distribution:

6a. $\forall x\ (\varphi(x) \wedge \psi(x)) \Leftrightarrow \forall x\ \varphi(x) \wedge \forall x\ \psi(x)$
6b. $\exists x\ (\varphi(x) \vee \psi(x)) \Leftrightarrow \exists x\ \varphi(x) \vee \exists x\ \psi(x)$

Prenex Laws (x does not occur as a free variable in ψ)

7a1. $\forall x\ (\varphi(x) \wedge \psi) \Leftrightarrow \forall x\ \varphi(x) \wedge \psi$
7a2. $\exists x\ (\varphi(x) \wedge \psi) \Leftrightarrow \exists x\ \varphi(x) \wedge \psi$
7b1. $\forall x\ (\varphi(x) \vee \psi) \Leftrightarrow \forall x\ \varphi(x) \vee \psi$
7b2. $\exists x\ (\varphi(x) \vee \psi) \Leftrightarrow \exists x\ \varphi(x) \vee \psi$
7c1. $\forall x\ (\varphi(x) \rightarrow \psi) \Leftrightarrow \exists x\ \varphi(x) \rightarrow \psi$ (! Quantifier changes!)
7c2. $\exists x\ (\varphi(x) \rightarrow \psi) \Leftrightarrow \forall x\ \varphi(x) \rightarrow \psi$ (! Quantifier changes!)
7d1. $\forall x\ (\psi \rightarrow \varphi(x)) \Leftrightarrow \psi \rightarrow \forall x\ \varphi(x)$
7d2. $\exists x\ (\psi \rightarrow \varphi(x)) \Leftrightarrow \psi \rightarrow \exists x\ \varphi(x)$